Modeling of the Plane Needle Cutting Edge Rake and Inclination Angles for Biopsy

Hollow needles are one of the most common medical devices, yet little study has focused on the needle tip cutting geometry for biopsy, which is a tissue cutting process. This research develops mathematical models to calculate the inclination and rake angles along cutting edges on needle tips generated by planes. Three types of plane needle tips, the one-plane bias bevel, multi-plane symmetrical, and two-plane nonsymmetric needles, are investigated. The models show that the leading tip of a bias bevel needle has an inclination angle of 0°, the worst configuration for cutting. Symmetric multiplane needles on the other hand have very high inclination angles, 60°, 56.3°, and 50.8° deg, given a needle formed by two-, three-, and four-plane, respectively, for a bevel angle of 30° and can assist more effective needle biopsy. The rake angle is at its greatest value (the best configuration for cutting), which equals the 90° deg minus the bevel angle, at the initial cutting point for the bias bevel needle. Experiments are performed using three 11 gauge two-plane symmetric needles with 20, 25, and 30 deg bevel angles on bovine liver and demonstrate that the needle tip geometry affects biopsy performance, where longer biopsy samples are collected with needles of higher rake and inclination angle. [DOI: 10.1115/1.4002190]

1 Introduction

Hollow needles are one of the most commonly used medical devices. A major medical procedure using hollow needles is biopsy, in which a tissue sample is cut and removed for examination to diagnose cancer or other diseases. There are a variety of different types of needle biopsy systems for cutting and retaining the tissue samples. One of which being commonly used is the end-cut needle biopsy, which is the focus of this paper. An end-cut biopsy is in general performed in three steps, as shown in Fig. 1. The needle and stylet (inner rod of needle) are first advanced to the front of the location where the biopsy is to take place (Fig. 1(a)); the needle is then advanced forward, cutting and trapping the tissue sample inside the needle (Fig. 1(b)); and lastly, the needle is removed from the body with the biopsy sample inside (Fig. 1(c)). All needle biopsy systems use the sharp edge of the needle tip to perform the primary cutting operation.

Current needle biopsy systems can be improved with longer biopsy sample lengths, decreases in sample fragmentation, and increases in biopsy performance consistency [1–3]. Studies have shown that the ability of doctors to make an accurate diagnosis is improved by biopsy samples that are longer and less fragmented [1,2]. Current end-cut biopsy needles have been shown to produce very inconsistent performance results, failing to capture any tissue 27% of the time [3].

The needle geometry has a direct affect on biopsy performance, as will be shown by biopsy experiments in Sec. 6. Blade cutting experiments performed by Moore et al. [4] also showed that cutting edge geometry of a blade plays an important role in the ability of the tissue to be cut in blade insertion experiments.

Needle tips come in a wide variety of different designs, yet there is a limited research on needle cutting geometry, the topic this paper directly addresses. The effects of needle geometry on insertion forces and needle deviation were analyzed by O’Leary et al. [5] and Podder et al. [6] for solid needles. Blade cutting force and modeling on tissue have also been studied before [7]. Needle bending analysis, needle steering, and tissue modeling have been performed in many studies [8–13]. This survey found no study on defining the cutting edge geometry of a biopsy needle. A better geometrical understanding of needle tips could lead to an optimized needle tip geometry, which would increase the biopsy sample length, decrease the amount of sample fragmentation, and increase the consistency of performance, thereby improving the doctor’s ability to make an accurate diagnosis.

Commercial needle tips are generated by a special burr-free grinding process. The simplest needle tip geometry is the one-plane bias bevel tip, as shown in Fig. 2. By indexing the needle tip in different orientations during grinding, various needle tip geometries can be generated by planes. Figure 3 shows examples of two-plane and three-plane needles. A multiplane needle contains planes oriented symmetrically or nonsymmetrically around the needle. A symmetric orientation means that the planes are tilted in equal angle relative to the needle axis and placed in equal intervals around the circumference of the cylindrical needle, while a nonsymmetric design means the planes are oriented in any other configuration. The two- and three-plane symmetric and nonsymmetric needle tips are illustrated in Fig. 3. Three parameters, the needle radius, the number of planes, and the tilt angle, are re-
required to define the symmetric plane needles. Defining the orientation for nonsymmetrical needles is more complex due to the added variables.

The curved leading edge of the needle tip acts as the cutting edge, as marked in Figs. 2 and 3, during the needle insertion into tissue. For the bias bevel needle, the cutting edge that performs the tissue cutting action during needle insertion transitions from the outer edge to the inner edge across the needle tip, as shown in Fig. 2(b). In the case of multiplane formed needles, the cutting edge is usually only the inner edge, as shown in Fig. 3. The performance of the needle for biopsy is dependent on the orientation and sharpness of cutting edge.

In machining, the tool geometry is critical to the cutting operation [14]. Two critical parameters at the oblique cutting edge that define the cutting geometry are the rake angle \( \alpha \) and inclination angle \( \lambda \) [15]. The rake and inclination angle affect the biopsy needle performance. Understanding the values of \( \alpha \) and \( \lambda \) is important for selecting the needle geometry for efficient tissue cutting in biopsy. The goal of this research is to develop mathematical models to calculate \( \alpha \) and \( \lambda \) around the needle circumference for various needle tips formed by planes. Needle insertion experiments are conducted to demonstrate the effects of \( \alpha \) and \( \lambda \) on biopsy cutting length. In this study, the model for inclination angle of a bias bevel needle tip is first developed, followed by symmetric multiplane needle tips, and then by nonsymmetric two-plane needle tips. The model for \( \alpha \) of the needle is then derived. The tool geometry parameters \( \alpha \) and \( \lambda \) are determined around the needle circumference. Specific factors in needle geometry influencing the \( \alpha \) and \( \lambda \) are also examined. Lastly, biopsy performance experiments are conducted, which measure the length of tissue biopsy sample and relates to the needle cutting edge geometry.

2 Mathematical Model for Inclination Angle of Plane Needles

2.1 Bias Bevel One-Plane Needle. The bias bevel needle is made by one-plane being ground at a specified bevel angle, \( \xi \), as shown in Fig. 4. An \( xyz \) axis with the \( z \) axis coinciding with the needle axis and the \( x \) axis passing through the lowest point of the needle tip profile is defined. The outside radius of the needle is \( r \) and the radial position of a point \( A \) along the needle is defined as \( y \).

When observing the bias bevel needle in the \( xz \) plane, as shown in Fig. 4(b), the needle profile forms a straight line with a slope of \( -\cot \xi \) and a \( z \) intercept of \( r \cot \xi \), thereby yielding the equation of \( z=(r-x)\cot \xi \). Putting in terms of \( \gamma \), the parametric equations of an ellipse to define the shape of the needle tip are

\[
\begin{align*}
x &= r \cos \gamma \\
y &= r \sin \gamma \\
z &= r(1 - \cos \gamma)\cot \xi
\end{align*}
\]

(1)

The normal vector of the \( xy \) plane \( \mathbf{v} = \{0,0,1\} \) and the tangent vector \( s \), shown at point \( A \) on Fig. 4(a), can be expressed as \( s = \{-r \sin \gamma, r \cos \gamma, r \cot \xi \sin \gamma\} \).

The angle between the \( P \) plane (plane with normal vector \( \mathbf{v} \)) and \( s \) is the inclination angle, \( \lambda \).

\[
\sin \lambda = \frac{|s \cdot \mathbf{v}|}{|s| |\mathbf{v}|} = \frac{|\cot \xi \sin \gamma|}{\sqrt{1 + \cot^2 \xi \sin^2 \gamma}}
\]

(2)

The equation for \( \lambda \) of one-plane bias bevel needle tip is

\[
\lambda(\xi, \gamma) = \arcsin \frac{|\cot \xi \sin \gamma|}{\sqrt{1 + \cot^2 \xi \sin^2 \gamma}} \quad (0 \leq \gamma \leq 2\pi)
\]

(3)

This inclination angle model along with all the other inclination and rake angle equations presented were validated by measuring the inclination and rake angles directly from SolidWorks models.

2.2 Symmetric Multiplane Needle. A symmetrical multiplane needle tip is formed by evenly spaced planes being positioned at identical bevel angles (\( \xi \)). Two-plane and three-plane needle tips are shown in Fig. 5. The \( xyz \) axis is defined using the same rule as in the bias bevel needle. Commercially, the three-
plane symmetric needle is called a Franseen needle.

The geometry of a symmetric multiplane needle is based on the same principles of using the parametric equations for an ellipse, Eq. (1). The inclination angle for a symmetrical multiplane needle is based on Eq. (3) with a phase shift of \((i-1)(2\pi/P)\) applied to \(\gamma\), where the \(P(\geq2)\) is an integer representing the number of planes and \(i=1,2,\ldots,P\) defines which segment of the plane is being examined. This leads to the general inclination angle equation for the symmetrical multiplane needle being

\[
\theta(\xi, P, \gamma) = \arcsin \left( \frac{\cot \xi \sin \left( \gamma - (i-1) \frac{2\pi}{P} \right)}{\sqrt{1 + \cot^2 \xi \sin^2 \left( \gamma - (i-1) \frac{2\pi}{P} \right)}} \right)
\]

\[
\frac{\pi}{P}(2i-3) \leq \gamma \leq \frac{\pi}{P}(2i-1)
\]

2.3 Nonsymmetric Two-Plane Needle. A nonsymmetric needle formed by two planes, as shown in Fig. 6, can be fully defined given the offset height between two planes \((h)\), offset angle between planes \((\psi)\), the bevel angle of the two planes relative to the \(z\) axis \((\xi_1\) and \(\xi_2)\), and the radius of the needle \((r)\). The needle cutting edge is formed by two ellipses, marked as ellipses I and II. These two ellipses intersect at two sharp tips. The non-symmetry of the planes makes determining the position of the sharp tip of the needle, i.e., location where the two planes meet, difficult. These two sharp points, marked as A and B in Fig. 6, have \(\gamma_A\) and \(\gamma_B\), respectively.

Based on Eq. (1), the parametric equations for ellipse I with bevel angle \(\xi_1\) are

\[
x_1(\gamma) = r \cos \gamma \\
y_1(\gamma) = r \sin \gamma \\
z_1(\gamma) = r(1 - \cos \gamma)\cot \xi_1
\]

The parametric equations for ellipse II can be formed by adding a phase shift \(\psi\) and the change in height \(h\).

\[
x_2(\gamma) = r \cos \gamma \\
y_2(\gamma) = r \sin \gamma \\
z_2(\gamma) = r(1 - \cos(\gamma + \psi))\cot \xi_2 + h
\]

The location of where the two planes meet is then found by setting \(z_1(\gamma) = z_2(\gamma)\).

\[
r(1 - \cos \gamma)\cot \xi_1 = r(1 - \cos(\gamma + \psi))\cot \xi_2 + h
\]

The Maple software is used to solve this equation in terms of \(\gamma_A\) and \(\gamma_B\), the location where two ellipses meet. The solution of \(\gamma_A\) and \(\gamma_B\) is long and presented in Appendix.

The inclination angle can be found at any \(\gamma\) along the radius of the needle for ellipses I,

\[
\theta(\xi_1, \gamma) = \arcsin \left( \frac{\cot \xi_1 \sin \gamma}{\sqrt{1 + \cot^2 \xi_1 \sin^2 \gamma}} \right) \quad (0 \leq \gamma \leq \gamma_A)
\]

and \((\gamma_B \leq \gamma \leq 2\pi)\)

and ellipse II,

\[
\theta(\xi_2, \psi, \gamma) = \arcsin \left( \frac{\cot \xi_2 \sin(\gamma + \psi)}{\sqrt{1 + \cot^2 \xi_2 \sin^2(\gamma + \psi)}} \right) \quad (\gamma_A \leq \gamma \leq \gamma_B)
\]

3 Mathematical Model for Rake Angle of Bias Bevel Needles

The oblique cutting configuration is applied to find the rake angle at a point on a bias bevel needle cutting edge. The rake angle \(\alpha\) is defined as the angle between two planes \(P_r\) and \(A\gamma\) measured in plane \(P_n\), as shown in Fig. 7. \(P_r\), is the plane at cutting point and parallel to the \(xy\) plane, \(A\gamma\) is the face plane of the needle tip surface, and \(P_n\) is a plane with a normal vector \(s\), vector tangent to cutting edge [14]. These three planes can be found on a bias bevel needle tip, as shown in Fig. 7. The normal vector to these planes are defined using \(\gamma, \xi,\) and \(r\). The plane \(P_n\) has a normal vector \(v = (0,0,1)\), the plane \(A\gamma\) has a normal vector \(n = \{\cos \xi_0, \sin \xi, 0\}\), and the plane \(P_r\) has a normal vector \(s = \{r \cos \gamma, r \sin \gamma, r \cos \xi \}\).

Vectors \(a\) and \(b\) mark the intersection of planes \(P_n\), \(A\gamma\), and \(P_r\), respectively, as illustrated in Fig. 7. Vectors \(a\) and \(b\) are defined using the cross products of the normal vectors:

\[
a = s \times n = \{r \cos \gamma \sin \xi, r \cos \xi \sin \gamma \cos \xi + r \sin \gamma \sin \xi, -r \cos \gamma \cos \xi\}
\]

\[
b = s \times v = \{r \cos \gamma, r \sin \gamma, 0\}
\]

The rake angle for a bias bevel needle is the angle between vector \(a\) and \(b\):
Equation (12) can also be applied to two-plane symmetric needles because the elliptical cutting edge of a bias bevel needle is identical to that of a two-plane symmetric needle.

4 Results of Inclination Angle and Discussions

4.1 Bias Bevel One-Plane Needle. Figure 8 shows the inclination angle for three bias bevel needle tips with $\xi = 15, 30, 45$ deg. At $\gamma = 0$ and 180 deg, $\lambda$ is equal to 0 deg, regardless of $\xi$. The tip of the bias bevel needle ($\gamma = 180$ deg), where the needle first contacts tissue during insertion, has $\lambda = 0$, which is the worst cutting configuration for needle insertion. This problem can be solved using the multiplane needle, as will be discussed in Sec. 4.2.

Smaller $\xi$ creates larger inclination angles where the maximum $\lambda = \pi/2 - \xi$ for a bias bevel needle. A larger $\xi$ reduces $\lambda$, thereby creating higher cutting forces [4].

4.2 Symmetric Multiplane Needle. A multiplane symmetric needle’s inclination angle can be solved using Eq. (4). The perspective view and $\lambda$ of a symmetric multiplane needle tip of $P = 2, 3, 4, \xi = 30$ deg, and $r = 1$ mm are shown in Fig. 9. The number of needle tip points (sharp points that occur where ellipses meet and equally space around the radius) on the needle tip is equal to $P$.

Similar to the bias bevel needle, as shown in Fig. 8, the minimum $\lambda$ is also equal to 0 for all symmetric multiplane needles. Unlike the bias bevel needle, the location of $\lambda = 0$ is the last point and not the first point on the needle tip cutting edge contacting the tissue. The number of points where $\lambda = 0$ is equal to the number of planes used to construct the needle because $\lambda = 0$ is located at the bottom of the ellipse formed by each plane.

At a tip of the multiplane needle where the needle first contacts tissue during insertion ($\gamma = 90$ deg for $P = 2$, $\gamma = 66.7$ deg for $P = 3$, and $\gamma = 45$ deg for $P = 4$), the $\lambda$ is maximum, which is the best cutting configuration for needle insertion. This is opposite to the bias bevel needle with $\lambda = 0$ at the needle tip point. The $\lambda = 60, 56.3, 50.8$ deg for $P = 2, 3, 4$, respectively, for $\xi = 30$ deg. For a given $\xi$, increase the $P$ beyond 2 reduces the maximum $\lambda$, thereby making the needle less effective at cutting.

The effect of $\xi$ on $\lambda$ of multiplane needles with $P = 2$ and 3 is shown in Fig. 10. The same trend as seen in the bias bevel needle (Fig. 8) is observed that the increase in $\xi$ reduces the $\lambda$.

4.3 Nonsymmetric Two-Plane Needle. The effects of $\psi$ and $h$ on $\lambda$ of nonsymmetric two-plane needles are discussed in the following sections.

4.3.1 Effect of $\psi$. Figure 11 shows the shape of the needle cutting edge and $\lambda$ for $\xi_1 = 30$ deg, $\xi_2 = 30$ deg, $r = 1$ mm, $h = 0$ mm, and $\psi = 45, 90, 180$ deg. Increasing $\psi$ shifts the location of the maximum and minimum $\lambda$, which occur on ellipse II. Ellipse I remains fixed in place, therefore, the $\lambda$ from $0 \leq \gamma \leq 90$ deg remains the same as $\psi$ varies from 45 to 180 deg. Changing $\psi$ can lower the high $\lambda$ that occurs on ellipse II, thereby making it more difficult to cut the soft tissue.
4.3.2 Effect of $h$. Figure 12 shows the shape of the needle cutting edge and inclination angle for $\xi_1 = 30$ deg, $\xi_2 = 30$ deg, $r = 1$ mm, $\psi = 135$ deg, and $l = 0.1$, and 2 mm. Increasing $h$ causes ellipses I and II to meet at $\gamma$ values closer to $\psi$. Again, ellipse I remains fixed in place; therefore, the $\lambda$ remains the same for 0 $\leq \gamma \leq 90$ deg. Varying $h$ can lower the maximum $\lambda$ reached by ellipse II.

A unique effect of varying $h$ is observed on $\lambda$, which becomes discontinuous as in the case of $h = 1.55$ and 2 mm. This occurs because the $\lambda$ on one side of the needle tip is different than the $\lambda$ on the opposite side of the needle tip. The cutting characteristics change greatly at this transition point. This interesting characteristic may greatly affect the cutting performance of the needle as a whole.

4.4 Comparison of the Inclination Angle. A bias bevel needle contains a full elliptical cutting edge profile that forms the base needle tip shape for all types of plane needles given they contain the same $\xi$ value. Adding more planes and varying the orientation of the planes only changes how the basic elliptical cutting edge profile is sectioned together and does not change the range of $\lambda$ that are possible for a given $\xi$ value.

The maximum $\lambda$ achieved by a plane needle is dictated by the plane with the smallest $\xi$, where $\lambda \leq \pi/2 - \xi$ for all plane needles. This occurs because the value $\pi/2 - \xi$ marks the steepest point on the bias bevel ellipse and therefore the point of maximum inclination. As illustrated in Fig. 9, this high $\lambda$ is not reached in the case of symmetric multiplane needles where $P > 2$. Varying the plane orientation, such as $h$ and $\psi$, can also determine if the maximum $\lambda$ is equal to $\pi/2 - \xi$.

The minimum $\lambda$ for all plane needles is 0 deg and the number of times that this occurs around the needle circumference is equal to $P$ given that $P > 1$. In the case of the one-plane bias bevel needle ($P = 1$), there are two points where $\lambda = 0$, at the top ($\gamma = 180$ deg) and bottom ($\gamma = 0$) of the elliptical cutting profile. Multiplane needles ($P > 1$) do not contain top portions of the elliptical cutting edge profile and instead contain one lower portion of the elliptical cutting edge profile for each plane that is used to form the needle tip. Thereby, creating the situation on multiplane needles where the number of points with $\lambda = 0$ equals the number of planes.

5 Results of Rake Angle and Discussion

Figure 13 shows the results of the $\alpha$ for a bias bevel needle with $\xi = 15, 30$, and 45 deg. The range of $\alpha$ is between 0 and $\pi/2 - \xi$. For other types of plane needles, this statement remains true because all plane needles are based on the same basic ellipse geometry.

The maximum $\alpha$ values occur at $\gamma = 0$ and 180 deg, top and bottom of the ellipse, with a value equal to $\pi/2 - \xi$. The higher rake angle represents a sharper cutting edge and makes it easier for tissue to pass over the needle cutting surface. All plane needles contain at least one relative minimum point of the ellipse; therefore, the maximum $\alpha$ is $\pi/2 - \xi$ for all plane-needles.

The minimum $\alpha$ occurs at $\gamma = 90$ and 270 deg, midpoints of the ellipse where the cutting edge changes from the inside to the outside of the needle, with a value of 0. The $\alpha$ does not become negative because the cutting edge changes sides at the point $\alpha = 0$. Varying $P$, $h$, and $\psi$ can potentially raise the minimum $\alpha$ if the midpoint of the ellipse is not revealed, thereby making the cutting edge always on the inside of the needle.
6 Biopsy Needle Experiments and Results

To study the effect of needle geometry on biopsy performance, biopsy experiments were performed on bovine liver tissue using 11 gauge two-plane symmetric needles of 20, 25, and 30 deg bevel angles. These needles are shown in Fig. 14. Two-plane symmetric needles are chosen because the symmetric tip design evenly directs tissue to the inside of the needle. A one-plane needle does not have this symmetry and therefore could cause the tissue to flow over the needle tip face rather than to the inside of the needle.

The overall biopsy experimental setup is shown in Fig. 15(a). The experimental setup uses a linear stage (Siskiyou Instrument model 200cri) to drive the needle into the tissue. One solid piece of fresh bovine liver is placed into a tissue holder with a cross section of \(50 \times 50 \text{ mm}^2\) and a pneumatic cylinder applies a constant pressure of 15.5 kPa to the top of the tissue to ensure repeatable tissue constraint conditions. The tissue holder has an open area where the needle is inserted into the tissue, as shown in Fig. 15(b). The linear stage inserts the needle through the 50 mm section of tissue at a constant speed of 1.5 mm/s. Once through the tissue, a stylet was used to push the biopsy sample out of the needle. The length of the biopsy sample out of the needle was then measured and recorded. Each of these three needles was inserted into the tissue ten times for a total of 30 trials.

Based on Eqs. (4) and (12), the complete range of inclination and rake angles of the three needles used in the experiments is shown in Fig. 16. The needles of higher bevel angles contain lower rake and inclination angles. Lower rake and inclination
angles have a negative effect on performance of the needles, as shown by the biopsy length results in Fig. 17. The needles with higher bevel angles collected shorter length samples of tissue. The experimental results demonstrate that the needle cutting edge geometry plays an important role in biopsy performance and the study of $\alpha$ and $\gamma$ around a needle is crucial to optimizing biopsy needle performance. It is worth noting that only a limited range of bevel angles, 20 deg $< \xi < 30$ deg, were tested. Outside this range, other factors may become dominated and affect the biopsy result.

7 Concluding Remarks

The inclination and rake angles around the radius of the needle were studied for the three needle tip styles examined: the bias bevel, symmetric multiplane, and nonsymmetric two-plane needles. The rake angle was solved for the bias bevel needle. The maximum inclination angle achievable is dictated by $\xi$ ($\lambda \leq \pi/2 - \xi$) and other variables such as $h$, $\psi$, and the number of planes could only decrease this maximum. For symmetric multiplane needles, increasing the number of planes lowered the inclination angle. For multiplane needles, the number of points where $\lambda=0$ was equal to the number of planes used to construct the needle. Varying $h$ and $\psi$ shifted the needle geometry, thereby changing where the ellipses met. The rake angle for plane needles was limited by $\xi$ ($0 \leq \alpha \leq \pi/2 - \xi$). Biopsy performance experiments show that the needle tip geometry defined by inclination and rake angles directly affects biopsy performance. Longer biopsy samples are obtained for two-plane needles with higher rake and inclination angles (lower bevel angles).

Future work will focus on two areas. One is to further study the needle geometry by examining needle geometries beyond the flat plane and by examining the other needle geometry characteristics, such as contact length and exposed needle area upon insertion. The second area is to more explicitly correlate the cutting parameters to the experimental performance of needle biopsy. New needle tip geometry will be manufactured and studied experimentally to determine the most efficient geometry for tissue biopsy cutting.

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Appendix: Solution of the $\gamma_A$ and $\gamma_B$ for the Two-Plane Nonsymmetric Needle

The solution of $\gamma_A$ and $\gamma_B$ are expressed related to two parameters for $E_1$ and $E_2$ with $0 < \psi < 180$ deg. In $E_1$, there is a minus sign in front of the square root while in $E_2$ there is a positive sign. If $\psi=180$ deg, then the limit as $\psi$ approaches 180 deg can be taken in order to find $\gamma_A$ and $\gamma_B$.

$$
\gamma_A = \arctan \left( \frac{r \tan \xi_1 - r \tan \xi_2 + h \tan \xi_2 \tan \xi_1}{-E_1 \tan \xi_1 \cos \psi} - \frac{\tan \xi_2}{\tan \xi_1 \cos \psi} + 1 \right)
$$

$$
\gamma_B = \arctan \left( \frac{r \tan \xi_1 - r \tan \xi_2 + h \tan \xi_2 \tan \xi_1}{-E_2 \tan \xi_1 \cos \psi} - \frac{\tan \xi_2}{\tan \xi_1 \cos \psi} + 1 \right)
$$

where

$$(E_1, E_2) = -h \tan^2 \xi_2 \tan \xi_1 + r \tan^2 \xi_1 \cos \psi - r \tan \xi_1 \tan \xi_2 + r \tan^2 \xi_2 - r \tan \xi_1 \cos \psi \tan \xi_2 + h \tan^2 \xi_1 \cos \psi \tan \xi_2 + 2r^2 \tan^2 \xi_1 \tan \xi_2 - 2r^2 \tan^2 \xi_1 \cos \psi \tan \xi_2 - 2r^2 \tan^3 \xi_1 \cos^3 \psi \tan \xi_2 - 2r^2 \tan^3 \xi_1 \cos^2 \psi \tan \xi_2 + 2r h \tan^3 \xi_1 \cos \psi \tan \xi_2 - 2r h \tan^3 \xi_1 \cos^2 \psi \tan \xi_2 + 2r h \tan^3 \xi_1 \cos^2 \psi \tan \xi_2 - 2r h \tan^3 \xi_1 \cos \psi \tan \xi_2$$

References


